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SOLVABILITY OF DIFFERENTIAL EQUATIONS WITH PERIODIC OPERATOR COEFFICIENTS USING GREEN'S FUNCTION

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ABSTRACT

In this paper, a Green's function has been derived with which a theorem is established that proves the uniqueness of the solution of a second order functional differential equation with periodic operator coefficients under some conditions on the unbounded operator. This operator has a domain and a range belonging to Hilbert's Space.

KEYWORDS: Functional Differential Equations, Periodic Operator Coefficients, Green's Function

MSC Number: 34K06, 34K13, 34K14

INTRODUCTION

Searching for periodic solutions for differential equations is not trivial. The main reason being that there are no general methods which may allow to establish if a periodic solution exists for some specific system of differential equations or not. Different methods and concepts should be inspected to find the best option but globally many of these methods are related to the perturbation theory (Nayfeh, 1973).

In applied mathematics and physics, second order differential equations or the equivalent system of two first order equations have a great importance (Boyce, 1986).

Many problems in physics and engineering lead to a system of linear differential equations with periodic coefficients. Lyapinov and Poincaré, who investigated the stability of periodic motions which are described by nonlinear differential equations transformed the centroid problem into a system of linear differential equations with periodic coefficients (Hale, 1993).

In the last years many results were achieved in the mathematical theory of differential equations with periodic coefficients, see (Benkhalti, 2004, Cabada, 2008, Huseynov, 2010, Kiguradze, 2009, Li, 2009, Nieto, 2005, Piao, 2004, Zhang, 2003).

Piao (2004) investigated the existence and uniqueness of periodic and almost periodic solution of the differential equation with reflection of argument. The relationship between modules of forced term and solution of the equation is considered.

Benkhalti and Ezzinbi (2004) studied the periodic solutions for some partial functional differential equations.

Li and Zhang (2009) dealt with the existence of positive T - periodic solutions for the damped differential equation

$\ddot{x}+p(t)\dot{x}+q(t)x = f(x,t)+c(t)$ where $p, q, c \in L^{-1}(\mathcal{R})$ are T - periodic functions and $f \in Car(\mathcal{R}\times\mathcal{R}^{+}, \mathcal{R})$ is T - periodic in the first variable.

According to Li's work, this proves that a weak repulsive singularity enables the achievement of new existence criteria through a basic application of Schauder's fixed point Theorem.

Huseynov (2010) investigated nonlinear second order differential equations subject to linear impulse conditions and periodic boundary conditions. Sign properties of an associated Green's function are exploited to get existence results for positive solutions of the nonlinear boundary value problem with impulse. The results obtained yield periodic positive solutions of the corresponding periodic impulsive nonlinear differential equation on the whole real axis.

Lillo (1968) indicates the extension of some of the results of Hahn (1961) for the Green's function to equations of the form considered by Shimanov (1963) for periodic differential difference equations. He also indicates the relation of this Green's function to the representation problem.

The results of Zverkin (1963) for the case of a scalar equation where the lags are multiples of the period are studied. Convergence result for the series associated with Green's function is established. This result along with those Lillo (1966) indicate a kind of “harmonic resonance” which occur in these equations.

Nieto (2005) obtained under suitable conditions, the Green's function to express the unique solution for a second-order functional differential equation with periodic boundary conditions and functional dependence given by a piecewise constant function. This expression is given in terms of the solutions for certain associated problems. The sign of the solution is determined taking into account the sign of that Green's function.

Zhang and Wang (2003) establish the existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order ordinary differential equations. The arguments are based only upon the positivity of the Green's functions and the Krasnoselskii fixed point theorem. They apply their results to a problem coming from the theory of nonlinear elasticity.

Cabada and Cid (2008) give a L^p-criterium for the positiveness of the Green's function of the periodic boundary value problem: $x'' + a(t)x = 0$, $x(0) = x(T)$, $x'(0) = x'(T)$ with an indefinite potential a(t). Moreover, they prove that such Green's function is negative provided $a(t)$ belongs to the image of a suitable periodic Ricatti type operator.

Theoretical Frame

Consider the second order equation:

$$L_p^2\, u(t) = D_t^2\, u(t) - \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj}\, M_{h_{kj}}\, D_t^k\, u(t) = f(t) \tag{1}$$

where A_{kj} are operators which domains belong to a Hilbert space X and their ranges to a Hilbert space Y, $X \subset Y$, $\|\cdot\|_X \geq \|\cdot\|_Y$ and A_{kj} : $Y \to Y$ are closed operators, A_{kj} : $X \to Y$ are bounded operators, $f(t)$ is ω - periodic function, h_{kj} are constants, $k = 0, 1$, $j = 0, \ldots, m$ and $h_{00} = h_{10} = 0$, $M_{h_{kj}}\, u(t) = u(t - h_{kj})$, $D_t^k = \frac{1}{i^k} \frac{d^k}{dt^k}$.

The existence of ω - periodic solutions of Eq. (1) is the main question in this work. For this, we consider the complete orthogonal system of functions $\left\{ e^{\frac{2i\pi nt}{\omega}}, n = 0, \pm 1, \cdots \right\}$ in $L_2(0, \omega)$ where $L_2(0, \omega)$ is Hilbert's space and we expand the function $f(t)$ in a series with this system, *i.e.*

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}} \tag{2}$$

Multiplying both sides of (2) by $e^{-\frac{2iknt}{\omega}}$ and integrating the resulting equation from 0 to ω, we get

$$\begin{aligned}\int_0^{\omega} f(t) e^{-\frac{2i\pi kt}{\omega}} dt &= \sum_{k=-\infty}^{\infty} f_n \int_0^{\omega} e^{\frac{2i\pi(-k+n)t}{\omega}} dt \\ &= \sum_{k=-\infty}^{\infty} f_n \begin{cases} \int_0^{\omega} dt, & n = k \\ 0, & n \neq k \end{cases} \\ &= \omega f_k .\end{aligned}$$

This implies that

$$f_k = \frac{1}{\omega}\int_0^{\omega} f(t) e^{-\frac{2i\pi kt}{\omega}} dt ,$$

Thus one can consider the expansion

$$f(t) \sim \sum_{n=-\infty}^{\infty} f_n e^{\frac{2i\pi nt}{\omega}} ,$$

$$f_n = \frac{1}{\omega}\int_0^{\omega} f(s) e^{-\frac{2i\pi ns}{\omega}} ds .$$

We will seek a periodic solution of (1) in the form of a Fourier series

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{\frac{2i\pi nt}{\omega}} , t \in \mathcal{R} .$$

Substituting $u(t)$, $f(t)$, $D_t^k u(t) = \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{2i\pi nt}{\omega}}$, $k = 0,1$, and

$$M_{h_{kj}} D_t^k u(t) = \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{-2i\pi n h_{kj}}{\omega}} e^{\frac{2i\pi nt}{\omega}}$$

into (1), we get

$$\begin{aligned}L_p^2 u(t) &= \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^2 e^{\frac{2i\pi nt}{\omega}} - \\ &\quad \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} \sum_{n=-\infty}^{\infty} u_n \left(\frac{2\pi n}{\omega}\right)^k e^{\frac{2i\pi n(t-h_{kj})}{\omega}} \\ &= f(t),\end{aligned}$$

$$L_p^2 u(t) = \sum_{n=-\infty}^{\infty} e^{\frac{2i\pi nt}{\omega}} \left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right] u_n$$

$$= \sum_{n=-\infty}^{\infty} f_n \, e^{\frac{2i\pi nt}{\omega}} ,$$

where E is the identity.

Equating the coefficients with the same powers of the exponential functions, we get

$$\left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right] u_n = f_n , \tag{3}$$

- Assuming that the following condition holds:

$$\left(\frac{2\pi n}{\omega} \right)^2 \in \rho(A_p) - \text{ resolvent set of the operator} \tag{4}$$

$$A_p = \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} M_{h_{kj}} D_t^k : X \to Y , \; n = 0, \pm 1, \cdots, \tag{5}$$

which means that the spectrum of the operator A_p does not contain the points of real axis

$$\left(\frac{2\pi n}{\omega} \right)^2 , \; n = 0, \pm 1, \cdots ,$$

- and from (3) we find

$$u_n = \left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right]^{-1} f_n . \tag{6}$$

If the equation

$$\left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right]^{-1} \varphi_n = 0. \tag{7}$$

Has a nontrivial solution $\varphi_0 \in X$, then the numbers $\left(\frac{2\pi n}{\omega} \right)^2$ belong to the spectrum of the operator A_p.

Introducing the notation

$$R_n = \left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right]^{-1} ,$$

equation (6) can be written in the form

$$u_n = R_n\, f_n \,, \tag{8}$$

which gives

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} u_n\, e^{\frac{2i\pi n t}{\omega}} \\ &= \sum_{n=-\infty}^{\infty} R_n\, f_n\, e^{\frac{2i\pi n t}{\omega}} \end{aligned} \tag{9}$$

where the resolvant operator $R_n : Y \to X$ depends on the parameter *n*. By virtue of the enclosure $X \subset Y$, , we can consider the operator $R_n : Y \to Y$.

Introducing in the right side of (9) the value of f_n, we get

$$u(t) = \int_0^{\omega} G(t-s)\, f(s)\, ds \,, \tag{10}$$

Where

$$G(t-s) = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} \left[\left(\frac{2\pi n}{\omega} \right)^2 E - \sum_{k=0}^{1} \sum_{j=1}^{m} A_{kj} \left(\frac{2\pi n}{\omega} \right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \right]^{-1} e^{\frac{2i\pi n (t-s)}{\omega}} \tag{11}$$

Or

$$G(t-s) = \frac{1}{\omega} \sum_{n=-\infty}^{\infty} R_n\, e^{\frac{2i\pi n (t-s)}{\omega}} \,.$$

We consider the function $\xi = \frac{t}{2}\left(\frac{t}{\omega} - 1 \right)$, $0 < t < \omega$. Since,

$$\begin{aligned} \xi_n &= \frac{1}{\omega} \int_0^{\omega} \left[\frac{t}{2}\left(\frac{t}{\omega} - 1 \right) \right] e^{-\frac{2i\pi n t}{\omega}}\, dt \\ &= \frac{1}{2\omega^2} \int_0^{\omega} t^2 e^{-\frac{2i\pi n t}{\omega}} - \frac{1}{2\omega} \int_0^{\omega} t\, e^{-\frac{2i\pi n t}{\omega}}\, dt \\ &= \frac{1}{2\omega^2} \left[t^2 \frac{-\omega}{2i\pi n} e^{-\frac{2i\pi n t}{\omega}} \Big|_0^{\omega} + \int_0^{\omega} 2t \frac{\omega}{2i\pi n} e^{-\frac{2i\pi n t}{\omega}}\, dt \right] \\ &\quad - \frac{1}{2\omega} \left[t \frac{-\omega}{2i\pi n} e^{-\frac{2i\pi n t}{\omega}} \Big|_0^{\omega} + \int_0^{\omega} \frac{\omega}{2i\pi n} e^{-\frac{2i\pi n t}{\omega}}\, dt \right] \\ &= \left(\frac{\omega}{2\pi n} \right)^2 \end{aligned}$$

and then subtracting from both sides of (11) the ω - periodic function $\xi(t)E$, expressed in uniformly convergent series ω - periodic functions, *i.e.*

$$\xi(t)E = \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}e^{\frac{2i\pi nt}{\omega}}E\ , \tag{12}$$

we get

$$\begin{aligned} G(t)-\xi(t)E &= \frac{1}{\omega}\sum_{n=-\infty}^{\infty}R_n\,e^{\frac{2i\pi nt}{\omega}} - \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}e^{\frac{2i\pi nt}{\omega}}E \\ &= -\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{n\neq 0}\left(\frac{1}{\omega}R_n - \frac{\omega}{(2\pi n)^2}E\right)e^{\frac{2i\pi nt}{\omega}} \end{aligned}$$

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left[\left(\frac{2\pi n}{\omega}\right)^2 R_n - E\right]e^{\frac{2i\pi nt}{\omega}}\ ,$$

adding and subtracting inside the square brackets of the operator

$$\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} \tag{13}$$

we get

$$\begin{aligned} G(t) &= \xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}\right.\right. \\ &\quad \left.\left. + \sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}\right]R_n - E\right)e^{\frac{2i\pi n t}{\omega}} \\ &= \xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\left[\left(\frac{2\pi n}{\omega}\right)^2 E - \sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}\right]R_n\right. \\ &\quad \left. + \sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n\, h_{kj}}{\omega}}R_n - E\right)e^{\frac{2i\pi n t}{\omega}} \end{aligned}$$

and finally

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}e^{\frac{2i\pi nt}{\omega}}R_n \tag{14}$$

Or

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1} + \sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}R_n\, e^{\frac{2i\pi n(t-h_{kj})}{\omega}} \tag{15}$$

For the series $\sum_{n \neq 0} \frac{\omega}{(2\pi n)^2} \left(\frac{2\pi n}{\omega} \right)^k R_n \, e^{\frac{2i\pi n (t - h_{kj})}{\omega}}$ the majorant will be the series

$$\sum_{n \neq 0} \frac{\omega}{(2\pi n)^2} \left(\frac{2\pi n}{\omega} \right)^k \left\| A_{kj} R_n \right\|_Y ,$$

Where

$$\left(\frac{2\pi n}{\omega} \right)^k \left\| A_{kj} R_n \right\|_Y = \left\| A_{kj} \left(\frac{2\pi}{\omega} \right)^k n^k R_n \right\|_Y \leq \begin{cases} c \left\| n^k R_n \right\|_X , & k = 0,1 \\ c \left\| n^k R_n \right\|_Y , & k = 2 \end{cases}$$

Hence, if we require that the conditions

$$\begin{aligned} &\left\| n^k R_n \right\|_X = O(1) , \quad k = 0,1 , \\ &\left\| n^k R_n \right\|_Y = O(1) , \end{aligned} \tag{16}$$

which are equivalent to $\left\| R_n \right\|_X = O\left(\frac{1}{\left| n^k \right|} \right)$, $k = 0,1$ and $\left\| R_n \right\|_Y = O\left(\frac{1}{\left| n^2 \right|} \right)$ must be held, then the series in (14) converges absolutely and uniformly and its sum is a continuous and periodic function.

The operator function $G(t)$ defined by (14) is called ω - periodic Green's function of (1) and it has the following properties:

- $G(t)$ is periodic: $G(t+\omega) = G(t)$.
- $G(t)$ is strongly continuous with respect to t and has strong derivatives except for the values $t = n\omega$, $n = 0, \pm 1, \ldots$. Moreover $G(+0) - G(-0) = 0$, $D_t G(+0) - D_t G(-0) = E$
- $G(t)$ has a second strong derivative and satisfy the equation $L_p^2 G(t) = 0$ with the defect mentioned in 2.

The first property immediately rises from the ω - periodicity of the exponential function $e^{\frac{2i\pi n t}{\omega}}$ and the function $\xi(t)$. Since $D_t \xi(t) = \frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{n} e^{\frac{2i\pi n t}{\omega}} = i \left(\frac{1}{2} - \frac{t}{\omega} \right)$, then it is easy to see that $D_t \xi(s+0) - D_t \xi(s-0) = 1$ that is

$D_t G(s+0) - D_t G(s-0) = E$ (property 2).

We now prove property 3. For this we re-write the expression for $G(t)$ in such a way that the series resulting after two differentiations is convergent.

$$G(t)=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}R_n e^{-\frac{2i\pi n(t-h_{kj})}{\omega}}$$

$$=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}e^{\frac{2i\pi n(t-h_{kj})}{\omega}}R_n\frac{\omega}{(2\pi n)^2}\left[\left(\frac{2\pi n}{\omega}\right)^k E-\right.$$

$$\left.\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}+\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}\right]$$

$$=\xi(t)E-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}$$

$$+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega^3}{(2\pi n)^4}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}e^{\frac{2i\pi n(t-h_{kj})}{\omega}}\left[\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}R_n+E\right].$$

From which we have

$$D_t G(t)=i\left(\frac{1}{2}-\frac{t}{\omega}\right)E+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega^2}{(2\pi n)^3}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}e^{\frac{2i\pi n(t-h_{kj})}{\omega}}$$
$$\left[\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}R_n+E\right]$$

and

$$D_t^2 G(t)=-\frac{E}{\omega}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}e^{\frac{2i\pi n(t-h_{kj})}{\omega}}$$
$$\left[\sum_{k=0}^{1}\sum_{j=1}^{m}A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}}R_n+E\right].$$

So,

$$G(t)=\frac{1}{\omega}+\sum_{n=-\infty}^{\infty}R_n e^{\frac{2i\pi n t}{\omega}}$$

$$=-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}+\frac{1}{\omega}\sum_{n\neq 0}R_n e^{\frac{2i\pi n t}{\omega}},$$

$$D_t^k G(t-h_{kj})=D_t^k\left[-\frac{1}{\omega}\left(\sum_{j=0}^{m}A_{0j}\right)^{-1}\right]+\frac{1}{\omega}\sum_{n\neq 0}\left(\frac{2\pi n}{\omega}\right)^k R_n e^{\frac{2i\pi n(t-h_{kj})}{\omega}},$$

$$\sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} D_t^k G\left(t-h_{kj}\right) = \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} D_t^k \left[\frac{-1}{\omega}\left(\sum_{j=0}^{m} A_{0j}\right)^{-1}\right]$$

$$+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{1}{\omega}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}\, e^{\frac{2i\pi n\left(t-h_{kj}\right)}{\omega}}\left(\frac{2\pi n}{\omega}\right)^2 R_n$$

$$=-\frac{E}{\omega}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^k A_{kj}\, e^{\frac{2i\pi n\left(t-h_{kj}\right)}{\omega}}\left(\frac{2\pi n}{\omega}\right)^2 R_n$$

$$=-\frac{E}{\omega}+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0}\frac{\omega}{(2\pi n)^2}\left(\frac{2\pi n}{\omega}\right)^2 A_{kj}\, e^{\frac{2i\pi n\left(t-h_{kj}\right)}{\omega}}$$

$$\left[E+\sum_{k=0}^{1}\sum_{j=1}^{m}\sum_{n\neq 0} A_{kj}\left(\frac{2\pi n}{\omega}\right)^k e^{-\frac{2i\pi n h_{kj}}{\omega}} R_n\right],$$

From the above expression, we have

$$D_t^2 G(t) = \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} D_t^k G\left(t-h_{kj}\right)$$

or

$$D_t^2 G(t) - \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} D_t^k G\left(t-h_{kj}\right) = 0\,,$$

That is, $G(t)$ is the solution of the homogeneous equation except for the points mentioned in property 2.

Using Green's function, we prove the following theorem.

Theorem

If

- the spectrum $\sigma(A_p)$ does not contain the real axis points $\frac{2\pi n}{\omega}$, $n = 0, \pm 1, \cdots,$
- the following condition holds:

$$\left(\frac{2\pi n}{\omega}\right)^2 \in \rho\left(A_p\right) - \text{ resolvent set of the operator}$$

$$A_p = \sum_{k=0}^{1}\sum_{j=1}^{m} A_{kj} M_{h_{kj}} D_t^k : X \to Y\,,\; n = 0, \pm 1, \cdots,$$

- and the condition (12) holds, then equation

$$L_p^2 u(t) = D_t^2 u(t) - \sum_{k=0}^{1}\sum_{j=0}^{m} A_{kj} M_{h_{kj}} D_t^k u(t) = f(t) \tag{17}$$

at any ω - periodic function $f(t)$, integrable and with bounded variations (Zygmund, 2002) has a unique solution, where A_{kj} are operators which domains belong to a Hilbert space X and their ranges to a Hilbert space Y, $X \subset Y$, $\|\cdot\|_X \geq \|\cdot\|_Y$ and $A_{kj} : Y \to Y$ are closed operators, $A_{kj} : X \to Y$ are bounded operators, $f(t)$ is ω - periodic function, h_{kj} are constants, $k = 0, 1$, $j = 0, \ldots, m$ and $h_{00} = h_{10} = 0$, $M_{h_{kj}} u(t) = u(t - h_{kj})$, $D_t^k = \frac{1}{i^k} \frac{d^k}{dt^k}$.

Proof

The existence of the integral in (17) comes from the strong continuity of the integrand. Hence, rewriting $u(t) = \int_0^t G(t-s) f(s) ds$, in the form:

$$u(t) = \int_0^t G(t-s) f(s) ds + \int_t^\omega G(t-s) f(s) ds ,$$

we have

$$\begin{aligned}
D_t^2 u(t) &= D_t^2 \int_0^\omega G(t-s) f(s) ds \\
&= D_t \left[\int_0^\omega G(t-s) f(s) ds + \left(G(+0) - G(-0)\right) f(t) \right] \\
&= D_t \left[\int_0^\omega G_t(t-s) f(s) ds \right] \\
&= \int_0^\omega G_{tt}(t-s) f(s) ds + \left(G_t(+0) - G_t(-0)\right) f(t) \\
&= \int_0^\omega G_{tt}(t-s) f(s) ds + f(t) \\
&= \int_0^\omega \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} D_t^k G(t - h_{kj} - s) f(s) ds + f(t) \\
&= \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} D_t^k \int_0^\omega G(t - h_{kj} - s) f(s) ds + f(t) \\
&= \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} D_t^k u(t - h_{kj}) + f(t)
\end{aligned}$$

That is:

$$D_t^2 u(t) - \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} D_t^k u(t - h_{kj}) = f(t) , \tag{18}$$

or

$$L_p^2 u(t) = f(t)$$

The uniqueness of solution results from the fact that in the theorem conditions the homogeneous equation cannot have nontrivial ω - periodic solutions.

CONCLUSIONS

We established a theorem concerning the uniqueness of the solution of the equation $L_p^2 u(t) = f(t)$ under some conditions on the operator

$$A_p = \sum_{k=0}^{1} \sum_{j=0}^{m} A_{kj} M_{h_{kj}} D_t^k : X \to Y .$$

A Green's function is built first and used afterwards to prove the theorem.

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